

Quantum holonomy in Lieb-Liniger model

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We examine a parametric cycle in the N -body Lieb-Liniger model that starts from the free system and goes through Tonks-Girardeau and super-Tonks-Girardeau regimes and comes back to the free system. We show the existence of exotic quantum holonomy, whose detailed workings are analysed with the specific sample of two- and three body systems. The classification of eigenstates based on clustering structure naturally emerges from the analysis.

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I. INTRODUCTION

Among the solvable models of quantum mechanics, Lieb-Liniger system [1] belongs to the selective class of models that are genuinely many-body. It is a system made up of identical bosons interacting through two-body contact force. It has later been shown that the one-dimensional system of identical fermions with two-body contact interactions can be rigorously mapped to Lieb-Liniger system with strong and weak coupling regimes interchanged [2, 3]. Several further extensions of the model with anyon statistics has been found [4–7], and they are also known to be mathematically equivalent to original model. Thermodynamics of Lieb-Liniger model has been extensively studied [8–11].

What has made the Lieb-Liniger model a focus of renewed recent attention is its experimental realisation in the form of Tonks-Girardeau gas [12–14]. It has been shown that the coupling strength of the Lieb-Liniger system can be experimentally controlled through the Feshbach resonance mechanism [15]. In recent experiments by Haller and collaborators [16, 17], a smooth change of the coupling strength from large negative values to large positive values, where one finds the super Tonks-Girardeau system [18], has been realised.

The continuous transition from a strongly repulsive to strongly attractive regimes of Lieb-Liniger model inspires us to propose following parametric cycle \mathcal{C} : We start with the non-interacting limit, increase the coupling strength adiabatically, reaches strongly attractive regime crossing the $\pm\infty$ coupling limit, then decreases the absolute value of negative coupling strength until it reaches the non-interacting limit again. In this paper, we show that the initial energy eigenstates of the cycle are different from the final eigenstates, although the initial and the final Hamiltonians are identical.

This phenomenon, the so-called *exotic* quantum holonomy, in which quantum eigenvalues and eigenstates do not come back to the original ones after a cyclic parameter variation [19], belongs to a wider class of quantum holonomy that comprises both the celebrated Berry phase [20] and the Wilczek-Zee holonomy [21] which appears in systems with degenerate eigenvalues. The exotic quantum holonomy in δ -function potential system has been already considered in [22]. Here we report a novel finding of the quantum holonomy in *many-body systems* interacting through δ -function potential.

The plan of this paper is as follows: In Sec. II, we derive the spectral equation for Lieb-Liniger model in two different forms to demonstrate the presence of quantum holonomies with respect to \mathcal{C} . In Sec. III, we show that the backward cycle is not always possible due to the clustering of particles. This leads to the concept of minimal states, which we utilize to classify the spectrum of the system in Sec. IV. In Sec. V, we

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provide another view of the quantum anholonomy by focusing on the two-body system through the complexification the coupling strength. Sec. VI contains our conclusion.

II. ADIABATIC CYCLE \mathcal{C} FOR LIEB-LINIGER MODEL

Let us consider N bosons confined in a one-dimensional space. The system is described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + g \sum_{j=1}^N \sum_{l=1}^{j-1} \delta(x_j - x_l), \quad (1)$$

where the unit is chosen such that \hbar and the mass of a particle can be set to unity. The parameter g is the interaction strength. We impose the periodic boundary condition to the position space with the period 2π .

We look at the dependence of eigenenergies and eigenvectors on the coupling strength g . In particular, we focus on the cycle \mathcal{C} , which consists of three stages $\mathcal{C}^{(s)}$ ($s = 1, 2, 3$): In the first stage $\mathcal{C}^{(1)}$, g is prepared to be 0 and, is adiabatically increased to ∞ ; Next, in the stage $\mathcal{C}^{(2)}$, g is suddenly flipped from ∞ to $-\infty$; In the final stage $\mathcal{C}^{(3)}$, g is again adiabatically increased to 0, which is the initial value of g . We denote the initial and final points of \mathcal{C} as $g = 0$ and $g = 0-$, respectively, to distinguish them.

The eigenvalue problem of H , Eq. (1) can be solved by the Bethe ansatz, where an eigenfunction is composed by N plane waves specified by a set of quasi-momenta, also called rapidity k_j , that satisfy

$$\exp(2\pi i k_j) = \prod_{l \neq j} \frac{k_{jl} + ig}{k_{jl} - ig}, \quad (2)$$

where $k_{ij} = k_i - k_j$ [1].

We examine how $k_j(g)$'s, which are chosen to be smooth as g is varied, are changed by the cycle \mathcal{C} . The function $k_j(g)$ completely characterizes the parametric evolution of eigenenergies as well as the “adiabatic” evolution of eigenvectors along \mathcal{C} . The analysis is decomposed into the three stages $\mathcal{C}^{(s)}$ ($s = 1, 2, 3$).

At the initial point $g = 0$ of the first stage $\mathcal{C}^{(1)}$, $k_j(0)$ takes an integer value. Without loss of generality, we

can choose the order of $k_j(g)$'s so as to satisfy $k_1(g) < k_2(g) < \dots < k_N(g)$ for small positive g [23]. This ensures $k_1(0) \leq k_2(0) \leq \dots \leq k_N(0)$.

We introduce two quantized quantities which is conserved during the parametric evolution of $k_j(g)$ along $\mathcal{C}^{(1)}$. Such “topological invariants” provide a way to evaluate the change of $k_j(g)$ induced by the stage $\mathcal{C}^{(1)}$.

Firstly, during the interval $0 \leq g < \infty$, we have an integer

$$I_j(g) \equiv k_j(g) - \frac{1}{\pi} \sum_{j \neq l} \arctan \frac{g}{k_{jl}(g)}. \quad (3)$$

This is a consequence of Eq. (2) and $(t+i)/(t-i) = -e^{-2i \arctan t}$, which is applicable as long as $t^{-1} \neq 0$. We use the principal branch of \arctan throughout this manuscript. This is justified for Eq. (3) because $g/k_{jl}(g)$ does not across its standard branch cuts, which emanate from $\pm i$ to $\pm i\infty$ [24]. The continuity and discreteness of $I_j(g)$ in $0 \leq g < \infty$ imply that $I_j(g)$ takes a constant value, which can be determined from the limit $g \downarrow 0$, i.e.,

$$I_j(g) = k_j(0). \quad (4)$$

Secondly, Eq. (2) and another formula for \arctan $(t+i)/(t-i) = e^{2i \arctan(t^{-1})}$, which holds for $t \neq 0$, implies that, in the interval $0 < g \leq \infty$,

$$J_j(g) \equiv k_j(g) + \frac{1}{\pi} \sum_{l \neq j} \arctan \frac{k_{il}(g)}{g} \quad (5)$$

is a half-integer for even N and an integer for odd N [1]. Following a similar argument for $I_j(g)$ above, we obtain the value of the invariant $J_j(g)$ for $0 < g \leq \infty$:

$$J_j(g) = k_j(\infty). \quad (6)$$

Now we evaluate the change of $k_j(g)$ during $\mathcal{C}^{(1)}$ using these invariants. From Eqs. (3) and (5), we obtain

$$k_j(\infty) - k_j(0) = \frac{1}{2} \sum_{l \neq j} \operatorname{sgn} \Re \frac{k_{jl}(g)}{g}, \quad (7)$$

where we used the identity

$$\arctan(t) + \arctan(1/t) = \frac{\pi}{2} \operatorname{sgn}[\Re(t)]. \quad (8)$$

We note that the right hand side of Eq. (7) makes sense only for $0 < g < \infty$. Here, $k_{jl}(g)$ is positive for

$j > l$ and negative for $j < l$, since we have assumed the order of $k_j(g)$ at the initial point of $\mathcal{C}^{(1)}$, and the sign of $k_{jl}(g)$ does not change for $g > 0$ [23]. This implies

$$\sum_{l \neq j} \text{sgn} \Re \frac{k_{jl}(g)}{g} = \sum_{l=1}^{j-1} - \sum_{l=j+1}^N. \quad (9)$$

Accordingly, we obtain

$$k_j(\infty) - k_j(0) = j - \frac{N+1}{2}. \quad (10)$$

Next we examine the second stage $\mathcal{C}^{(2)}$, where g suddenly changes from ∞ to $-\infty$. Note that all $k_j(\infty)$'s are finite because of Eq. (10). Since a finite root of the Bethe equation, Eq. (2), at $g = \infty$ is also its root at $g = -\infty$, we employ a smooth extension of $k_j(g)$ along $\mathcal{C}^{(2)}$, i.e.,

$$k_j(-\infty) = k_j(\infty). \quad (11)$$

The analysis for the final stage $\mathcal{C}^{(3)}$ resembles to the one for $\mathcal{C}^{(1)}$. First, $J_j(g)$ is a half-integer for even N and an integer for odd N in the interval $-\infty \leq g < 0$. In order to ensure that $J_j(g)$ is an invariant within this interval, we need to assume that k_{jl}/g does not cross the branch cuts of arctan. The proof of this assumption is to be published elsewhere [25]. Here let us just note that it is true for $N = 2$, and is numerically confirmed for a numbers of lower-lying states of $N = 3$ and 4 in the following sections. Second, $I_j(g)$ is also an integer for $-\infty < g \leq 0$. We also assume that g/k_{jl} does not cross the branch cut of arctan in this interval. This implies $k_j(0-) = I_j(g)$. The change of $k_j(g)$ in the path $\mathcal{C}^{(3)}$ is given by

$$k_j(0-) - k_j(-\infty) = -\frac{1}{2} \sum_{l \neq j} \text{sgn} \Re \frac{k_{jl}(g)}{g}. \quad (12)$$

We can ensure that

$$\Re[k_1(g)] < \Re[k_2(g)] < \dots < \Re[k_N(g)], \quad (13)$$

because it holds at $g = -\infty$. Recalling the fact that g is negative here, we obtain

$$k_j(0-) - k_j(-\infty) = j - \frac{N+1}{2}. \quad (14)$$

Combining above three arguments, we obtain a non-trivial change of $k_j(g)$ due to \mathcal{C} in the form

$$k_j(0-) - k_j(0) = 2j - (N+1). \quad (15)$$

Note that the total momentum remains unchanged during the cycle \mathcal{C} . The final energy and state after the adiabatic cycle, however, are different from the initial ones, showing that \mathcal{C} induces the eigenenergy and eigenspace anholonomies [22]. We also remark that $k_1 < k_2 < \dots < k_N$ holds at the end of \mathcal{C} . This implies that we can repeat the adiabatic cycle \mathcal{C} arbitrarily, and, the repetition of \mathcal{C} will induce the further instances of the eigenenergy and eigenspace anholonomies.

We can summarize our results in terms of a mapping between two sets of quasi-momenta of free bosons, i.e., $k_j(0)$'s and $k_j(0-)$'s. It is sufficient to consider the case that initial condition $n_j \equiv k_j(0)$ satisfies $n_1 \leq n_2 \leq \dots \leq n_N$. With the notation $n'_j \equiv k_j(0-)$, the mapping $(n_1, n_2, \dots, n_N) \mapsto (n'_1, n'_2, \dots, n'_N) = F(n_1, n_2, \dots, n_N)$, which is given by

$$\begin{aligned} F(n_1, n_2, \dots, n_N) \\ = (n_1 - N + 1, n_2 - N + 3, \dots, n_N + N - 1), \end{aligned} \quad (16)$$

expresses the quantum holonomy induced by the cycle \mathcal{C} .

III. INVERSE CYCLE

We now examine the inverse of the cycle \mathcal{C} . In contrast to the forward cycle \mathcal{C} , the parametric variation along the inverse \mathcal{C}^{-1} is not always possible. This is because the clustering of particles at $g = -\infty$ induces the divergence of eigenenergy [1]. Such a clustering invalidates the use of the Hamiltonian, Eq. (1). We call an eigenstate of free boson at $g = 0$ a *minimal state* if the the parametric variation along \mathcal{C}^{-1} is impossible. The precise condition for appearance of the minimal state is the subject of this section.

Formally, \mathcal{C}^{-1} corresponds to the inverse of the mapping F , Eq. (16), on the sets of quasi-momenta at $g = 0$:

$$\begin{aligned} F^{-1}(n_1, n_2, \dots, n_N) \\ = (n_1 + N - 1, n_2 + N - 3, \dots, n_N - N + 1), \end{aligned} \quad (17)$$

where we impose the ordering condition $n_1 \leq n_2 \leq \dots \leq n_N$. When the distance between n_j 's are far enough, F^{-1} preserves the ordering. This is the case

that \mathcal{C}^{-1} can be realized, and the resultant energy and quantum state are the solution of the eigenvalue problem of H , Eq. (1), at $g = 0$. On the other hand, when a pair of n_j s is too close, F^{-1} breaks the ordering, which implies the emergence of the clustering of particles during the inverse cycle. There are two possible cases. The first case is where a pair of quasi-momenta, say, n_j and n_{j+1} are degenerate, i.e., $n_j = n_{j+1}$. By applying F^{-1} , the resultant quasi-momenta satisfy $n_j > n_{j+1}$. In fact, the eigenenergy diverges $-\infty$ as $g \rightarrow -\infty$ during \mathcal{C}^{-1} . The second case, $n_j = n_{j+1} + 1$, also leads the clustering of particles.

The argument above is sufficient to determine the condition for the minimal states. When there is, at least, a pair of two quasi-momenta at $g = 0$ that satisfies

$$|n_j - n_{j+1}| \leq 1, \quad (18)$$

states specified by n_j and n_{j+1} are minimal states.

IV. CLASSIFICATION OF SPECTRA

Because of the existence of quantum holonomy, some states are reachable by the repetitions of parametric cycles \mathcal{C} and \mathcal{C}^{-1} starting from one particular eigenstate, while other states are not. This offers the classification of whole eigenstates into families of states connected by quantum holonomy. Such a family can be specified by a minimal state introduced above, because an arbitrary eigenstate with a finite energy can become minimal by a finite repetition of \mathcal{C}^{-1} .

From one minimal state, we can find other minimal states using the symmetries of the Hamiltonian (1). Suppose that a minimal state is specified by quasi-momenta (n_1, n_2, \dots, n_N) . The translational symmetry implies that $(n_1 + 1, n_2 + 1, \dots, n_N + 1)$ is also a minimal state, whose total momentum is larger by N than the original one. For an arbitrary integer ℓ , $(n_1 + \ell, n_2 + \ell, \dots, n_N + \ell)$ is also a minimal state. The reflection symmetry implies that $(-n_N, \dots, -n_2, -n_1)$ is also a minimal state, which may or may not be different from the original state.

Hence, it is sufficient to find all minimal states

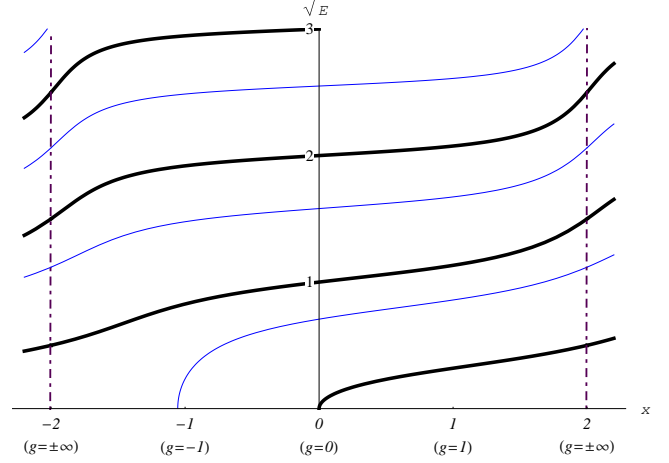


FIG. 1. Parametric evolution of eigenenergies of two-body Lieb-Liniger model, where x and y axes indicate $(4/\pi) \arctan g$ and \sqrt{E} , respectively. The thick (black) and thin (blue) lines correspond to the families specified by the minimal states $(0, 0)$ and $(0, 1)$, respectively. See Eqs. (21) and (22). Note that the eigenenergies are continuous at $g = \pm\infty$.

whose total momenta satisfy the condition

$$-\frac{N}{2} < \sum_j n_j \leq \frac{N}{2}, \quad (19)$$

to enumerate all minimal states using the translational symmetry, offering a way to classify the spectra of the Lieb-Liniger model completely. We illustrate this classification for a few-body cases.

We start the analysis of $N = 2$ case with two minimal states

$$(0, 0) \quad \text{and} \quad (0, 1). \quad (20)$$

We obtain two families of eigenstates at $g = 0$ from these two minimal states, by repeating \mathcal{C}

$$(0, 0) \mapsto (-1, 1) \mapsto (-2, 2) \mapsto \dots, \quad (21)$$

and

$$(0, 1) \mapsto (-1, 2) \mapsto (-2, 3) \mapsto \dots, \quad (22)$$

respectively. The eigenenergies of these families are depicted in Fig. 1. By shifting the total momentum from the two minimal states, Eq. (20), we obtain an infinite number of minimal states (ℓ, ℓ) and $(\ell, \ell + 1)$

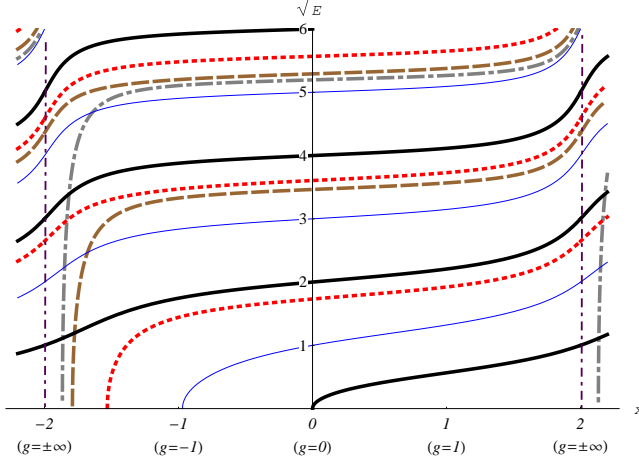


FIG. 2. (Color online) Eigenenergies of $N = 3$ families, where x and y axes are the same as in Fig. 1. The total momentum of all families shown here is zero. The thick (black) line corresponds to the family (Eq. (24)) specified by the minimal state $(0, 0, 0)$. The thin (blue) line corresponds to the $(-1, 0, 1)$ -family. These two families are trimers. The dotted (red), dashed (brown) and dashed-dotted (gray) lines are dimer families specified by minimal states $(-1, -1, 2)$, $(-2, -2, 4)$ and $(-3, -3, 6)$ respectively. Although there are level crossings, the adiabatic theorem ensures that the adiabatic time evolution is confined within a family [33].

with an arbitrary integer ℓ . The (ℓ, ℓ) - and $(\ell, \ell + 1)$ -families have the set of quasi-momenta at $g = 0$ given by $\{(\ell - m, \ell + m)\}_{m=0}^{\infty}$ and $\{(\ell - m, \ell + 1 + m)\}_{m=0}^{\infty}$, respectively. This exhausts the minimal states and families for $N = 2$.

The $N = 3$ case is far more complex than the $N = 2$ case. First, we consider the case that the total momentum is zero, where an infinite number of minimal states can be found. We depict some of them in Fig. 2. There are two minimal states

$$(0, 0, 0) \quad \text{and} \quad (-1, 0, 1), \quad (23)$$

which are called trimers [26], because the clustering of all three particles occurs in the limit $g \rightarrow -\infty$. The family of eigenstates at $g = 0$ specified by the minimal state $(0, 0, 0)$ is:

$$(0, 0, 0) \mapsto (-2, 0, 2) \mapsto (-4, 0, 4) \mapsto \dots \quad (24)$$

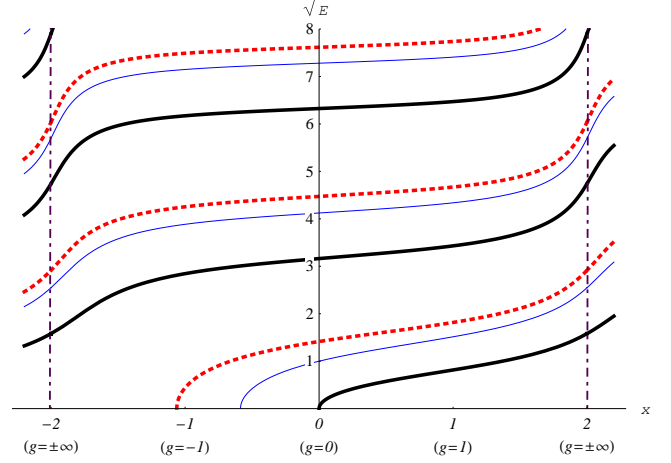


FIG. 3. (Color online) Parametric evolution of eigenenergies of the $N = 4$ case, where x and y axes are the same as in Fig. 1. The thick (black) line corresponds to the family: $(0, 0, 0, 0) \mapsto (-3, -1, 1, 3) \mapsto (-6, -2, 2, 6) \dots$. The thin (blue) and dotted (red) lines correspond to $(-1, 0, 0, 1)$ - and $(-1, -1, 1, 1)$ -families, respectively.

Besides, there are an infinite number of minimal states

$$\{(-\ell, -\ell, 2\ell)\}_{\ell>0} \quad \text{and} \quad \{(-2\ell, \ell, \ell)\}_{\ell>0}, \quad (25)$$

where the latter set can be induced through the use of the reflection symmetry. These minimal states are called dimers [26], because the clustering of two particles occurs in the limit $g \rightarrow -\infty$. Second, we consider the case $\sum_j n_j = 1$. We have a trimer

$$(0, 0, 1), \quad (26)$$

and an infinite number of dimers

$$\begin{aligned} \{(-\ell, -\ell, 2\ell + 1)\}_{\ell>0}, \quad \{(-\ell, -\ell + 1, 2\ell)\}_{\ell>0}, \\ \{(-2\ell + 1, \ell, \ell)\}_{\ell>0}, \quad \{(-2\ell, \ell, \ell + 1)\}_{\ell>0}. \end{aligned} \quad (27)$$

Note that all minimal states that satisfy $\sum_j n_j = -1$ can be obtained from the minimal state with $\sum_j n_j = 1$ through the use of the reflection symmetry. We obtain all other minimal states from above using the translational and reflection symmetry.

It is possible to enumerate minimal states and associated spectral families in a similar way for larger N . We simply close this section by showing several families of $N = 4$ system in Fig. 3.

V. EXCEPTIONAL POINTS

So far we have focused on the quantum holonomy induced by the real cycle \mathcal{C} . In Ref [27], it is argued that, through an analysis of a quantum kicked top, the quantum holonomy has a correspondence with the exceptional points, where the Hamiltonian exhibits non-Hermitian degeneracy [28, 29]. In other words, it is conjectured that the eigenenergy and eigenspace anholonomy can be understood as a result of the metamorphosis of eigenenergies and eigenstates induced by the encirclements around the exceptional points. In the following, we offer another example of this conjecture using the two-body Lieb-Liniger model by deforming \mathcal{C} in the complexified g -space.

The analysis requires the non-Hermitian extension of the Lieb-Liniger model [30]. We obtain eigenenergies with complex-valued coupling parameter g through numerical computation. We here focus on $(0,0)$ -family (Eq. (21)). Let $E_n(g)$ denote the eigenenergy of the state whose quasi-momenta take $(-n, n)$ at $g = 0$. We depict $E_n(g)$ for $n = 0, 1, 2$ in Fig 4. We find that these eigenenergies compose a Riemann surface. Its Riemann sheets $E_n(g)$ are connected by the exceptional points and associated branch cuts (cf. Ref. [31]).

Under the present choice of the branch cuts, all exceptional points of $(0,0)$ -family appear in the $E_0(g)$ sheet. A pair of eigenenergies $E_n(g)$ ($n > 0$) and $E_0(g)$ has a pair of degenerate points g_n and g_n^* , where we choose $\Im g_n < 0$. We find that all degenerate points are of degree two. Hence the pair of eigenenergies for an exceptional point exhibits square-root type singularity. The encirclement around the exceptional point g_n in complex g -plane induces the permutation of $E_0(g)$ and $E_n(g)$. We numerically confirm these properties of g_n with $n = 1, 2, \dots, 10$. We find that $\Re g_n$ and $\Im g_n$ are decrease monotonically as n increases. We also obtain similar result for $(0,1)$ -family.

Let us consider the cycle that is a concatenation of \mathcal{C} and C_1 in Fig 4 (d). Because this cycle encircles the exceptional point g_1 , the cyclic permutation (E_0, E_1) occurs. On the other hand, the cycle composed by \mathcal{C} and C_2 induces the cyclic permutation among (E_0, E_1, E_2) . As the cycle involves more deeper

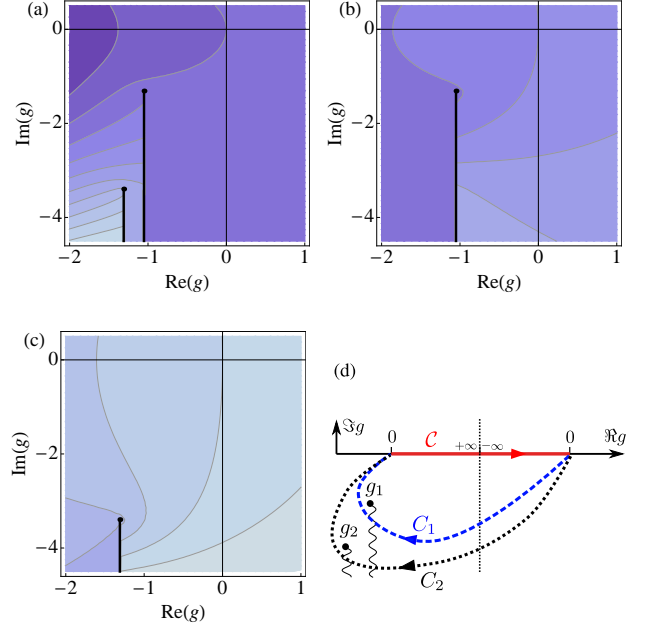


FIG. 4. (Color online) Contour plots of $\Re E_n(c)$: (a) $n = 0$; (b) $n = 1$; (c) $n = 2$. Lighter (darker) color indicates larger (smaller) value of $\Re E$. Thin lines are the contours of $\Re E_n(c)$. The exceptional points are indicated by filled circles. Bold lines indicate the branch cuts. While all complex EPs appear in $E_0(c)$, each $E_n(c)$ ($n = 1, 2$) has a single EP. (d) Schematic explanation of complex cycles that enclose exceptional points. We depict \mathcal{C} by a thick (red) line. Dashed (blue) and dotted (black) curves indicate C_1 and C_2 , respectively. See the main text.

exceptional points, the accuracy of the the resultant permutation become better to approximate a shift to eigenenergies $(E_0, E_1, \dots) \mapsto (E_1, E_2, \dots)$, which is realized by the quantum holonomy along the cycle \mathcal{C} . In this sense, we may say that the spectrum of Lieb-Liniger model feels the exceptional points that reside in the complex parameter space to induce the quantum holonomy along \mathcal{C} .

VI. CONCLUSION

We have shown, in this work, that an eigenstate of the free Lieb-Liniger system $g = +0$ is transformed to another eigenstate with higher energy in the process of eigenspace anholonomy involving the parametric cycle $g : +0 \rightarrow +\infty : -\infty \rightarrow -0$. Experimental testing should be within the range of current techniques [16, 17]. On the way to prove the existence of quantum

holonomy, we have demonstrated that the eigenstates of the Lieb-Liniger model can be classified according to their clustering property. The two and three boson system have been analysed in detail.

Our result can be interpreted in terms of geometry. Consisting of real numbers and $\pm\infty$, the parameter space of coupling strength is homeomorphic to S^1 . Therefore, our anholonomy is affected by the topology of S^1 . The presence of two kinds of invariants $I_j(g)$, Eq. (3), and $J_j(g)$, Eq. (5), for the parametric evolution of $k_j(g)$ reflects the fact that at least two charts are required for S^1 . Converting one of spectrum condition to the other by using the formula of arctan (8) corresponds to coordinate transformation.

The cycle of winding number m , \mathcal{C}^m , increases $k_j(0)$ by $m[2j - (N + 1)]$.

The topological nature of the the quantum holonomy implies that it is stable against, at least, small perturbations [32]. This also suggests that an experimental realization of the quantum anholonomy is possible in one-dimensional bosonic systems.

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